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(6) OPTIMAL INVENTORY POLICY,

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P-189

Revised
16 November 1950

The RAND Corporation
SANTA MONICA · CALIFORNIA

5 September 1950
Revised 16 November 1950
P-189

1. Introduction

1:A We propose to outline a method for deriving optimal rules for inventory policy. The problem of inventories exists not only for business enterprises but also for non-profit agencies such as governmental establishments and their various branches. Moreover, the concept of inventories can be generalized so as to include not only goods but also disposable reserves of manpower as well as various stand-by devices. Also, while inventories of finished goods present the simplest problem, the concept can be extended to goods which can be transformed, at a cost, into one or more kinds of finished goods if and when the need for such goods arises. The following notes prepare the way for a more general analysis of "flexible planning."

1:B We shall make explicit use of the concepts of probability and utility. Probabilities enter because at least some of those variables that are not controlled by the policymaker are random variables. Of these, the quantity demanded (from him) per unit of time is the principal one. Some other variables, such as the purchase price, or the time required for the fulfillment of an order ("pipeline time") may also be random, but will not be considered as such in the present paper.

1:C If the future were certain, the policymaker would determine the variables under his control — for example, the amounts he orders — in such a way as to maximize a certain quantity, called net utility (or minimize its negative, called net damage, or net loss). In enterprises run for profit, the total money profit earned over a long period of time (with future profits possibly converted into present values by using a time-discount factor based on some appropriate rate of interest) is a convenient measure of net utility. Accordingly, most of the existing writings on the inventory control of business firms make explicit use of the notion of maximum profit. These writings, though often not too clear in presentation and not developed in detail, do provide, in essence, a satisfactory solution for the case when all variables not controlled by the firm are known in advance with certainty.

1:D The natural extension to the case when at least some of the non-controlled variables are random variables, is to maximize the expected (actuarial, average) value of profit, or of utility. This presupposes the explicit use of the probability distribution of demand (or, more generally, the joint probability distribution of demand, pipeline time, purchase price, and other non-controlled variables). This distribution either can be known in advance, or may have to be estimated with the necessary precision as the data on demand, etc., are being accumulated.

1:E The random nature of the non-controlled variables is recognized implicitly in the customary provision for "cushion" or "safety-margin" stocks. However, we have not been able to find in the business literature an explicit rule of determining "cushion stocks" that would maximize expected profit (or minimize expected loss), given the relevant probability distribution.

1:F Explicit use of a (Poisson) distribution of demand was made, in a pathbreaking manner, by Thornton C. Fry [6]. This was later developed by Churchill Eisenhart [3] and apparently also applied by R. H. Wilson [10]. In this approach, the recommended rules of action are derived not by prescribing that the expected net utility be a maximum but by prescribing that the probability of stock depletion should have a certain level. This is analogous to the fixing of a "significance level" by a hypotheses-testing statistician of the pre-Wald era. (In various writings since 1939, Abraham Wald has suggested choosing statistical test-procedures so as to minimize the expected loss to the policymaker.) The choice of the suitable probability of stock depletion must ultimately depend on utility considerations, albeit in a hidden fashion. We shall try to make such considerations explicit. (See also 3:E below.)

1:G In "non-profit" organizations utilities other than money must be used. As for uncertainty, it is of course always present in non-profit no less than in commercial organizations. Various organizations have their own rules for taking care of uncertainty. However, it is not always obvious how these rules were derived from considerations of utility (e.g., the loss caused by the inability of a supplying agency to meet an urgent requirement) and from considerations of probability (e.g., the probability that a requirement will not be met).

1:H To sum up our own approach: the net utility to any policymaker is, in general, a random variable depending on certain conditions (i.e., on variables or on relations between variables). Some of these conditions he can control, others he cannot. The former are policy means (strategies). The non-controlled conditions are, in general, defined by a joint probability distribution of certain variables. Rational policy consists in fixing the controlled conditions so as to maximize the expected value of net utility, given the probability distribution of non-controlled conditions. When this probability distribution degenerates into a set of non-random variables we have the case of "certainty." In this limiting case, net utility itself is a non-random variable, to be maximized by the policymaker.

1:I As already mentioned (in 1:B), at most one of the non-controlled conditions will be regarded in the present paper as a random one: the rate of demand for the policymaker's product. Other non-controlled conditions will be regarded as constants, or as relations with constant parameters: the relation between storage cost and the size of industry; the relation between purchasing price and the size of order ("supply function"); and the cost of making an order.

As to controlled conditions, we shall assume that the policymaker can control only the size of the orders he makes. This eliminates, for example, such policy means as the fixing of the selling price, or the use of advertising, to influence demand; and any bargaining with buyer(s) or competitor(s).

Although it would be interesting and useful to broaden the problem in the various directions just indicated, we believe our specialized formulation is a workable first approximation. By regarding the order size as the only controlled condition, and the demand as the only random non-controlled condition, we do take account of most of the major questions that have actually arisen in the practice of business and nonprofit organizations.*

1:J Section 2 of the present paper will give the essentials of the optimal stock determination under conditions of certainty. The remaining sections will treat the uncertainty case, considering demand as the only random variable. Section 3 discusses a static model. Section 4 formulates the mathematical problem for a simplified dynamic model, for which Section 5 outlines a method of solution. Section 6 contains examples: solutions for the simplified dynamic model are given, assuming specific distributions of demand. Possible extensions of this model are briefly defined in Section 7.

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*Before formulating the problem, a study was made of the existing business literature on inventory control, using freely the comprehensive bibliography [10] that was compiled by T. H. Whitin and Louise B. Haack for the Logistics Research Project of the Office of Naval Research at the George Washington University. Some of the above suggestions, broadening the problem so as to embrace the models of perfect and imperfect markets as discussed in the academic economic theory, are due to Mr. Markowitz, Cowles Commission for Research in Economics.

2. The Case of Certainty

.2:A Let x be the known constant rate of demand for the product of the organization, per unit of time. Let the gross utility (i.e., utility before deducting cost) obtained by the organization through satisfying this demand, be

$$ax + a_0 .$$

We can assume $a_0 = 0$ (this will not influence the solution of our problem). In this case, if the organization is a commercial firm, $\underline{a}$ is the selling price; otherwise $\underline{a}$ is the value to the organization of an operation of some kind. In general, $\underline{a}$ is a function of x; but it will be sufficient, for our purposes, to assume $\underline{a}$ constant. Denote by b the purchasing price (in money units or utility units) of one unit. Because of the possible economy of large scale orders, b is a non-increasing function of the amount ordered: $b = b(S)$, $b' \leq 0$, assuming differentiability. Let K be the cost of handling an order, regardless of its size. Let z be the stock, and let the cost of carrying it over one unit of time be

$$\text{const.} + 2cz ,$$

the constant part being the overhead cost of storage. In general, the coefficient c may be a function of z, but it will suffice here to assume c constant.

.2:B Assume, to begin with, that orders are fulfilled immediately. Then orders must be made whenever the stock reaches zero: to order earlier would cause unnecessary storage cost, and to order later would cause unsatisfied demand. The amount ordered must equal the maximum stock, S. The average level of stock will be S/2; see Figure 1, where θ is the time elapsing

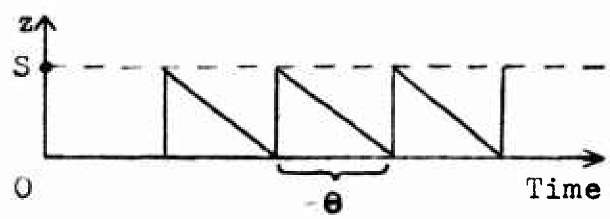

Fig. 1

between making an order and having the stock completely exhausted. Clearly S (or θ) is controlled by the policymaker, and

(2.1) $$S = x\theta .$$

We shall first assume that θ can take all real positive values (but see 2:E). The problem is to choose the optimal S and θ. Consider a time interval of θ units, beginning with the ordering of S units of goods and ending when the stock is exhausted. The total utility derived during this period, apart from a constant overhead cost of storage,* is

$$aS - bS - 2c\theta\left(\frac{S}{2}\right) - K ,$$

*We neglect the fact that a part of the overhead cost of storage — the interest and the amortization charge on stored goods — depends on the purchasing price b(S); the required modification is simple.

or, per unit of time, by (2.1),

$$(2.2) \qquad ax - bx - cS - (xK/S) .$$

Suppose the future time over which utility is maximized — the "horizon" — is either infinite or is long enough to be considered, without great error, as a sequence of an integral number of periods, of θ units each, with initial stocks at zero. Then the maximizing of utility over the "horizon" with respect to S is equivalent to the maximizing of (2.2), or (since $\underline{a}$ is independent of S) to minimizing the cost

$$(2.3) \qquad C = xb(S) + cS + (xK/S) .$$

Hence the optimal value of the amount ordered, $S = S^*$, must satisfy the equation

$$(2.4) \qquad xb'(S^*) + c = xK/(S^*)^2 ,$$

and the inequality

$$(2.5) \qquad b''(S^*) > -2[c + xb'(S^*)]/S^* .$$

2:C We shall assume the purchase price function (supply function) b(S) linear, so that $b'' = 0$ identically and

$$(2.6) \qquad b(S) = b_0 - b_1 S \quad \text{(say)}$$

with $b_1 \geq 0$. Then by (2.1), (2.3)

$$(2.7) \qquad C = b_0 x + x(c - b_1 x)\theta + K/\theta = C(\theta), \quad \text{say.}$$

The optimal re-ordering period $\theta = \theta^*$ is

$$(2.8) \qquad \theta^* = \sqrt{K/x(c - b_1 x)} \ .$$

Note that for θ^* to be real and finite, the following constraint upon the demand, and upon the conditions of storing and purchasing, must be satisfied:

$$0 \leq b_1 < c/x \ .$$

The minimum cost, C^*, is by (2.7), (2.8)

$$(2.9) \qquad C^* = C(\theta^*) = b_0 x + 2\sqrt{Kd} \ ,$$

where

$$(2.10) \qquad d = x(c - b_1 x) > 0,$$

and the positive root is taken. The optimal size of an order is by (2.8), (2.1),

$$(2.11) \qquad S^* = \sqrt{Kx/(c - b_1 x)} \ .$$

Hence, as should be expected, the optimal order size, and therefore the optimal ordering interval is the larger, the larger the cost K of handling an order, the smaller the unit storage cost c and the larger the effect b_1 of the size of order upon the unit price.

We believe this is, in essence, the solution advanced by R. H. Wilson [10], formerly of the Bell Telephone Company; and also by other writers; see [1].

2:D If we now introduce a constant "pipeline time," $\tau > 0$, elapsing between order and delivery, this will not affect S^* or θ^*, but the time of issuing the order will be shifted τ time units ahead. The order will be issued when the stock is reduced, not to zero, but to $x\tau$ units.

2:E The policymaker may not have full control of the length of the time interval between any two successive orders. Transportation schedules or considerations of administrative convenience may be such as to make ordering impossible at intervals of length other than, say, $\theta^0 \neq \theta^*$. For example, θ^0 may be one business day or week; or θ^0 may be the period between two visits of a mail-boat to an island depot. Orders of positive size may be spaced only at intervals of length $\theta = m\theta^0$, where m is a positive integer. We have to find the value $m = m'$ that will minimize the cost (2.3). The corresponding optimal length of interval between two non-zero orders will be denoted by θ', and the optimal order size by S'. The optimal set (m', θ', S') is unique since, as previously indicated in 2:B, it can never pay to possess a non-zero stock at a time when reordering takes place (or, more generally, τ time units after the reordering, $\tau > 0$; see 2:D). Hence, at a reordering time, the situation is always the same as that at the beginning of the entire process. Therefore, there is no possibility of gaining by having a variable θ' or S'.

To find the optimal values for m, θ, S, note that (2.7) is a continuous function of θ, with a unique minimum at $\theta = \theta^*$ when θ is not restricted.

Suppose first that the permissible interval length $\theta^0 > \theta^*$, and therefore the interval length between the successive non-zero orders, $\theta = m\theta^0 > m\theta^*$. Then (2.7) is smallest when $m = m' = 1$. Hence $\theta' = \theta^0$, $S' = x\theta^0$. For example, if $\theta^* = 14$ days but orders can be placed only every $\theta^0 = 30$ days, then optimal reordering period $\theta' = 30$ days (not 60 or 90, etc.). Suppose next that $\theta^0 < \theta^*$, and that the ratio $n = \theta^*/\theta^0$ is an integer, $n > 1$. Then obviously (2.3) is minimized at $\theta' = \theta^* = n\theta^0$; that is, $m' = n$, $S' = S^*$. (For example, if $\theta^* = 14$, $\theta^0 = 7$ days, then orders will be issued every $\theta' = 14$ days.) Finally, if $\theta^0 < \theta^*$ but the ratio n is not an integer, consider the integer $\bar{n}$, $\bar{n} < n < \bar{n} + 1$. Then $m' = \bar{n}$ or $\bar{n} + 1$, i.e., the optimal length, θ', of the interval between two successive non-zero orders will be either $\bar{n}\theta^*$ or $(\bar{n} + 1)\theta^*$, whichever leads to a smaller cost (2.3). (For example, if $\theta^* = 11$, $\theta^0 = 7$, then $\theta' = 7$ or 14.)

2:F As an important generalization of the case of fixed intervals between (positive or zero) successive orders, one would have to consider the case when orders can be made at any time, but at varying handling cost K. Let $K = K^0$ for the instants 0, θ^0, $2\theta^0$, $\cdots$, and $K = K^+ > K^0$ for all other instants. (This degenerates into the case of sub-section 2:E when K^+ is infinite.)

Let us refer to periods which are multiples of θ^0 as scheduled periods, the others as non-scheduled periods. If we consider only the latter, then by (2.9), the minimum attainable cost is $b_0x + 2\sqrt{K^+d}$. For scheduled periods the cost, by (2.7), (2.10), is $b_0x + d\theta + (K^0/\theta)$. The range of values of θ for which the last expression is not greater than the minimum cost attainable with a non-scheduled period is, then,

$$d\theta + (K^o/\theta) \leq 2\sqrt{K^+ d} \ ,$$

or,

$$d\theta^2 - 2\sqrt{K^+ d}\ \theta + K^o \leq 0,$$

or,

$$(2.12) \qquad d^{-\frac{1}{2}}\left(\sqrt{K^+} - \sqrt{K^+ - K^o}\right) \leq \theta \leq d^{-\frac{1}{2}}\left(\sqrt{K^+} + \sqrt{K^+ - K^o}\right) .$$

Therefore, a scheduled period will be used if a multiple of θ^o falls in the interval (2.12); otherwise, the optimal non-scheduled period, $\sqrt{K^+/d}$, is used. Replacing θ by $n\theta^o$ in (2.12) and dividing through by θ^o yields the following condition for using a scheduled period: The interval from

$$(2.13) \qquad d^{-\frac{1}{2}}\left(\sqrt{K^+} - \sqrt{K^+ - K^o}\right)/\theta^o \quad \text{to} \quad d^{-\frac{1}{2}}\left(\sqrt{K^+} + \sqrt{K^+ - K^o}\right)/\theta^o$$

should contain a positive integer.

Clearly, a sufficient condition is that the interval (2.13) contain the number 1, while a necessary condition is that the upper limit should be at least 1. Suppose the upper limit is at least 1 and at the same time at least twice the lower limit. Then either 1 belongs to the interval (2.13), in which case a scheduled period should be used, or 1 lies below the lower limit. Let n be the largest integer below the lower limit of (2.13); then the integer 2n lies below the upper limit and above the lower, so that again the interval (2.13) contains a positive integer.

The condition that the upper limit be at least twice the lower is that

$$2\left(\sqrt{K^+} - \sqrt{K^+ - K^0}\right) \leq \sqrt{K^+} + \sqrt{K^+ - K^0} \; ,$$

which is equivalent to $K^+/K^0 \geq 9/8$. If

$$(2.14) \qquad K^+/K^0 \geq 9/8 \text{ and } \sqrt{K^+} + \sqrt{K^+ - K^0} \geq \theta^0 d^{\frac{1}{2}} \; ,$$

then a scheduled interval should be used.

(2.14) is a sufficient condition while (2.13) is both necessary and sufficient.

2:G In this and the next two sub-sections, we conclude the discussion of the case of a certainty by remarks on the problem of "aggregation." Let there be several commodities, numbered $i = 1, 2, \cdots$, and characterized by, generally, different storage cost coefficients c_i and different purchase price functions b_i. Let us first assume that the cost, K, of handling an order does not depend on the size or composition of the order. The problem is to find optimal ordering intervals for the several commodities, possibly arranging the commodities into subsets so that all members of a subset are ordered simultaneously.

Consider the set of the first q commodities. Assuming a linear purchase price function for each commodity and applying the notations of sub-section 2:C, with a commodity subscript i where necessary, compare the following two costs (per unit of time): 1) the minimum cost, $\sum_{i=1}^{q} C_i^*$ of ordering, buying and storing the q commodities, when each commodity is reordered at intervals of length θ_i^* chosen so as to minimize C_i; 2) the minimum cost $\bar{C}$ of ordering, buying and storing the commodities when all are ordered at the same time at intervals of length $\bar{\theta}$. We have by (2.8), (2.9), (2.10),

$$(2.15) \qquad \theta_i^* = \sqrt{K/d_i}\ , \qquad d_i = x_i(c_i - b_{1i}x_i) > 0, \qquad i = 1, \cdots, q,$$

$$(2.16) \qquad \sum C_i^* = \sum b_{0i}x_i + 2 \sum \sqrt{Kd_i}\ ,$$

where the summation is from 1 to q, and every root is positive. On the other hand, $\theta = \bar{\theta}$ minimizes the expression

$$\sum b_{0i}x_i + \theta \sum d_i + K/\theta\ ,$$

analogous to the right-hand side of (2.7), with $\sum d_i$ replacing d. Hence

$$(2.17) \qquad \bar{\theta} = \sqrt{K/\sum d_i}\ ;$$

$$(2.18) \qquad \bar{C} = \sum b_{0i}x_i + 2\sqrt{\sum Kd_i}\ ,$$

the root being positive. $\bar{C}$ is always smaller than $\sum C_i^*$, since the square of $\sum \sqrt{Kd_i}$ exceeds the square of $\sqrt{\sum Kd_i}$ by

$$2 \sum_i \sum_j K \sqrt{d_i d_j}\ , \qquad i \neq j,$$

which is positive. Thus, if K does not depend on the composition of the order, it is preferable to order all q commodities of the considered subset at the same time, provided the common period length is determined as in (2.17). This is true for any q, and therefore also for the set of all commodities.

2:H The situation becomes different if the cost of handling the order depends on its composition. Let, for example, K(i) be the cost of handling an order (of any size) for the i^{th} commodity, K(i, j) the cost of handling an order for any quantities of commodities i and j, etc. Even if we still maintain the assumption that $K(1) = K(2) = \cdots = K$, the advantage of aggregation may disappear if we do not maintain any more that, for the given set of q commodities, also $K = K(1, 2, \cdots, q)$. (For example, the handling of an order requiring the services of an aircraft expert as well as a canned food specialist may be much more expensive than the ordering of these commodities separately.) We have, then, in fact, to compare $\sqrt{K} \sum \sqrt{d_i}$ with $\sqrt{K(1, \cdots, q)} \sqrt{\sum d_i}$, or $K\left(\sum \sqrt{d_i}\right)^2$ with $K(1, 2, \cdots, q) \sum d_i$. Aggregation is advantageous if the right-hand expression is smaller than the corresponding left-hand one, i.e., when

$$(2.18) \qquad \frac{K(1, 2, \cdots, q)}{K} < \frac{\left(\sum_{i=1}^{q} \sqrt{d_i}\right)^2}{\sum_{i=1}^{q} d_i} .$$

In particular, if $d_i = d_j = d$ for all i, j, the above condition becomes

$$\frac{K(1, 2, \cdots, q)}{K} < \frac{q^2 d}{qd} = q ;$$

i.e., it is required that the cost of ordering the q commodities separately should be less than q times the cost of ordering them jointly. In the general case $d_i \gtreqless d_j$, $i, j = 1, \cdots, q$, the condition (2.18) can be interpreted as follows. Define $e_i = \sqrt{d_i}$. Then

$$(2.19) \qquad \sum_i d_i = \sum_i e_i^2 = q(\sigma_e^2 + \bar{e}^2) ,$$

where σ_e is the standard deviation of the e's and $\bar{e}$ their mean. Also,

$$(2.20) \qquad \left(\sum_i \sqrt{d_i}\right)^2 = \left(\sum_i e_i\right)^2 = q^2 \bar{e}^2 .$$

From (2.18), (2.19) and (2.20), the condition that aggregation be advantageous is that

$$(2.21) \qquad \frac{K(1, 2, \cdots, q)}{K} < \frac{q^2 \bar{e}^2}{q(\sigma_e^2 + \bar{e}^2)} = \frac{q}{V^2 + 1}$$

where $V = \sigma_e / \bar{e}$ is the coefficient of variation of the e's. The expression $1/(V^2 + 1)$ indicates the dissimilarity between the q considered commodities with respect to their storage cost and purchase conditions. Condition (2.21) states that aggregation is advantageous if the joint cost of ordering the set of q commodities, multiplied by a "cost-dissimilarity index," is less than q times the cost of ordering any one commodity separately.

2:I A problem analogous to that of optimal aggregation of commodities into groups, is that of the optimal number of storage depots; or more generally, the optimal (bivariate) distribution of orders among the givers of orders (depots) and the receivers of orders (manufacturers and transporters). This presupposes the knowledge of the storage cost -- c_h say -- of the h^{th} order-giver and the knowledge of the big-lot price reduction -- b_{1k} say -- of the k^{th} order-receiver; these result in a joint frequency distribution of a parameter d_{hk}, analogous to the parameter d_i in (2.15). Some fundamentals of this problem, given certain parameters of utility and cost, were treated by Tompkins in [8].

3. A Static Model with Uncertainty

3:A Suppose an organization wants to choose the level $z \geq 0$ that the stock of a certain commodity should have at the beginning of a given period, in order to provide for the demand (requirements) that will occur during that period. We shall choose the time unit to be equal to the length of this period, and use the notations of Section 2. Thus $x \geq 0$ will denote the demand during the period. However, x will now be regarded as a random variable. We shall suppose that the organization knows the cumulative distribution of demand $F(x)$ (but see 7:D below). The utility, to the organization, of delivering ξ units of commodity will be

$$(3.1) \qquad a\xi + a_0 , \qquad (\underline{\underline{a}} \text{ constant}).$$

The delivery during the period is a random variable: ξ equals x or z, whichever is smaller. Hence the expected utility derived from satisfied demand is

$$(3.2) \qquad az[1 - F(z)] + a\int_0^z x \, dF(x) + a_0 .$$

We shall assume that the amount to be spent in purchasing ξ units is

$$(3.3) \qquad \xi(b_0 - b_1 \xi) + K; \quad b_0 > 0, \quad b_1 \geq 0;$$

so that as in Section 2, the purchase price is either constant or linearly decreasing with the amount purchased. As before, the cost of handling an order is denoted by K but this term will not play any further role in the static model. However, we assume here that the whole stock z is to be purchased (so that always $\zeta = z$), and that no utility is derived from satisfying demand after the period's end. Finally, the cost of carrying over our period the stock which has level z at the beginning of the period, will be assumed to be

$$\text{const.} + cz \,. \tag{3.4}$$

Then, apart from a "depletion penalty" which we shall introduce in 3:B, the net expected loss (the negative of net expected utility) is

$$\text{const.} + z(c + b_0 - b_1 z) - az[1 - F(z)] - a\int_0^z x \, dF(x) \,. \tag{3.5}$$

3:B We now define Π, the "depletion penalty," as follows: if $x \leq z$, there is no unsatisfied demand, and $\Pi = 0$; but if $x > z$, the organization would be willing to pay an amount $\Pi > 0$ to satisfy the excess, $x - z$, of demand over available stock.

We assume the penalty function as given. The organization — whether commercial or noncommercial — has a general idea of the value it would attach to the damage that would be caused by the non-availability of an item; it knows the cost and the poorer performance of emergency substitutes. The penalty for depleted stocks may be very high: "A horse, a horse, my kingdom for a horse," cried defeated Richard III.

3:C Note that, in the case of a commercial enterprise, an independent penalty function $\Pi = \Pi(x - z)$ need not be introduced. It can be replaced by considerations on "losing custom," as in the following model. Let F_t be a Poisson distribution of demand for the period $(t, t+1)$. Its mean, μ_t, is proportional to the probability that a member of a large but finite reservoir of customers will want to buy during that period. μ_t equals μ_{t-1} if the demand during $(t-1, t)$ was satisfied. But if that demand was in excess of the then available stock, μ_t is smaller than μ_{t-1}, by an amount proportional to the unsatisfied demand, as some of the disappointed customers will drop out of the market. The problem is to maximize total expected utility over a sequence of periods $(0, 1)$, $(1, 2)$, $\cdots$, if the initial distribution F_0 is given. Such a dynamic model would be more complicated than the one we are going to treat in Sections 4–7.

3:D We shall assume

$$\Pi = A + B(x - z), \quad \text{if } x > z,$$

$$\Pi = 0 \quad \text{otherwise},$$

where A, B are non-negative constants, not both zero. Then Π is a random variable, with expectation

$$(A - Bz)[1 - F(z)] + B\int_z^\infty x \, dF(x) \,. \tag{3.6}$$

Accordingly, the expected net loss, taking account of expected penalty, is the sum of the expressions (3.5) and (3.6) and equals, apart from a constant,

$$z(c + b_0 - b_1 z) + A[1 - F(z)] - (B + a)z[1 - F(z)] - (B + a)\int_0^z x \, dF(x) \tag{3.7}$$

$$= L(z) \,,$$

say. The stock level $z = z^*$ is optimal if $L(z^*) \leq L(z)$ for every z. Suppose the distribution function $F(x)$ possesses a differentiable density function $f(x) \equiv dF(x)/dx$. If the absolute minimum of L is not at $z = 0$, it will be at some point satisfying the relations

$$dL(z^*)/dz = 0, \quad d^2L(z^*)/dz^2 > 0,$$

which imply that

$$[c + b_0 - 2b_1 z^*] - Af(z^*) - (B + a)[1 - F(z^*)] = 0, \tag{3.8}$$

$$-2b_1 - Af'(z^*) + (B + a)f(z^*) > 0\,. \tag{3.9}$$

3:E In the economist's language, the first bracketed term in (3.8) is the "marginal cost" (of buying and carrying an additional unit in stock); the remaining two terms yield the "marginal expected utility."

It is seen from (3.8) that the optimal stock z^* is determined by the following "physical data," or "non-controlled parameters": 1) the demand distribution function $F(x)$; 2) certain utility and cost parameters: $(c + b_0)$, b_1, A, and $(B + a)$. If, in particular, $b_1 = 0$ (i.e., the economy of big-lot purchases is negligible), these parameters reduce to two: $A/(c + b_0)$ and $(B + a)/(c + b_0)$. To simplify further, for the sake of illustration, suppose also that $B = a = 0$: that is, the penalty is either zero or A, independent of the size of the unsatisfied demand; and utility derived from the functioning of the organization does not depend on the amounts delivered. Then (3.8), (3.9) become

(3.10) $\qquad f(z^*) = (c + b_0)/A; \quad f'(z^*) < 0.$

A graphical solution for this case is shown in Figure 2. (Note that $f'(z^*) < 0$ but $f'(z') > 0$; z^* is the best stock level, but z' is not.)

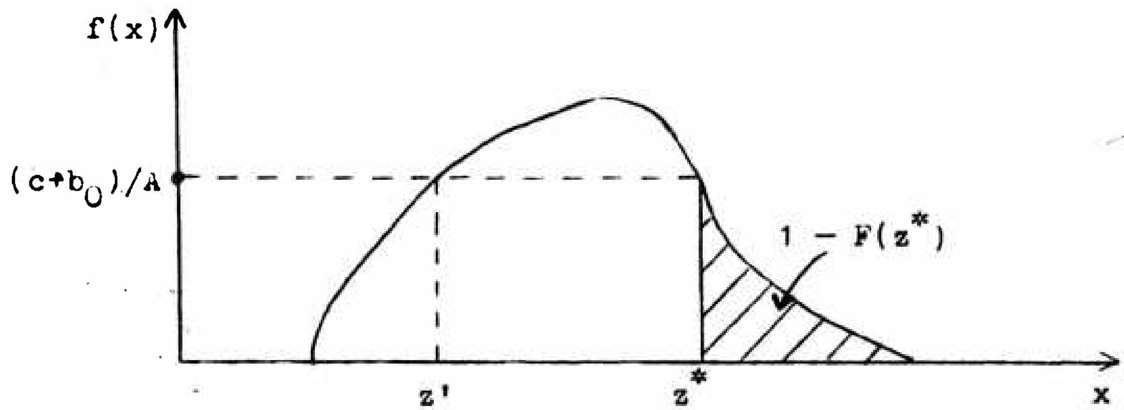

Figure 2

3:F In some previous literature (Ref. [6, 3]), the decision on inventories was related, not to utility and cost considerations, but to a preassigned probability $(1 - F(z))$ that demand will not exceed stock. The choice of the probability level $1 - F(z)$ depends, of course, on some implicit evaluations of the damage that would be incurred if one would be unable to satisfy demand. In the present paper, these evaluations are made explicit. On the other hand, since the value of the parameters such as A, B, $\underline{a}$ can be estimated only in a broad way (at least outside of a purely commercial organization, where utility = dollar profit, and where models such as that of 3:C can be developed), it is a welcome support of one's judgment, to check these estimates by referring to the corresponding level of probability for stock depletion. For example, if the distribution on Figure 1 were approximately normal, then to assume that penalty A is 100 times the marginal cost $c + b_0$ would be equivalent to prescribing that the shaded area measuring the depletion probability should be 0.3%; to assume that $A = 10 (c + b_0)$ would be equivalent to making depletion probability = 5%, etc.

3:G In the more general case, when $B + a > 0$ (but still $b_1 = 0$), a given optimal stock level z^*, and consequently a given probability of depletion $1 - F(z^*)$ is consistent with a continuous set of values of the pair of parameters: $A/(c + b_0) = A'$, $(B + a)/(c + b_0) = B'$, such as would satisfy the linear equation (3.8). For example, if $F(x)$ is normal, then an optimal stock exceeding the average demand by two standard deviations of demand (and, consequently, a depletion probability of 2.3%), will be required by any pair of values of A', B' lying on the straight line intersecting the A' -- axis at 13 and intersecting the B' -- axis at 44; while an optimal stock exceeding the average demand by three standard deviations (and, consequently, a depletion probability of 0.1%), will correspond to a straight line intersecting those axes at 228 and 740, respectively. Thus a set of contour lines helps to choose an interval of optimal stock values consistent with a given region of plausible values of utility-and-penalty parameters.

4. A Dynamic Model of Uncertainty: Problem

4:A The model described in Section 3 may be called a "static" one. We shall now present a "dynamic" one. In this model, the commodity can be ordered, and reordered, at discrete instants $0, \theta_0, \cdots, t\theta_0, \cdots$, where θ_0 is a fixed constant (but see 7:B). We can therefore use θ_0 as a time unit. Let x_t be the demand over the interval $(t, t+1)$. Assume the probability distribution of demand $F(x)$ to be independent of t. Denote by y_t the stock available at instant t, not including any replenishment that may arrive at this instant. Denote by z_t the stock at t including the replenishment. Denote by o_t the amount ordered at time t. Let the time between the ordering and the receiving of goods ("pipeline time") be τ. Then,

$$y_t = \max(z_{t-1} - x_{t-1}, 0), \qquad t = 1, 2, \cdots, \tag{4.1}$$

$$z_{t+\tau} = y_{t+\tau} + o_t, \qquad t = 0, 1, \cdots. \tag{4.2}$$

In general, τ is a non-negative random variable. We shall, however, assume $\tau = 0$ to simplify the analysis at this stage. Then (4.2) becomes

$$z_t = y_t + o_t . \tag{4.3}$$

Choose two numbers S and s, $S > s > 0$, and let them define the following rule of action:

$$\begin{aligned} &\text{If } y_t > s, \quad o_t = 0 \quad (\text{and hence } z_t = y_t), \\ &\text{if } y_t \leq s, \quad o_t = S - y_t \quad (\text{and hence } z_t = S). \end{aligned} \tag{4.4}$$

Figure 3 shows the sort of curve that might be obtained for stock level as a function of time if such a rule is adopted.

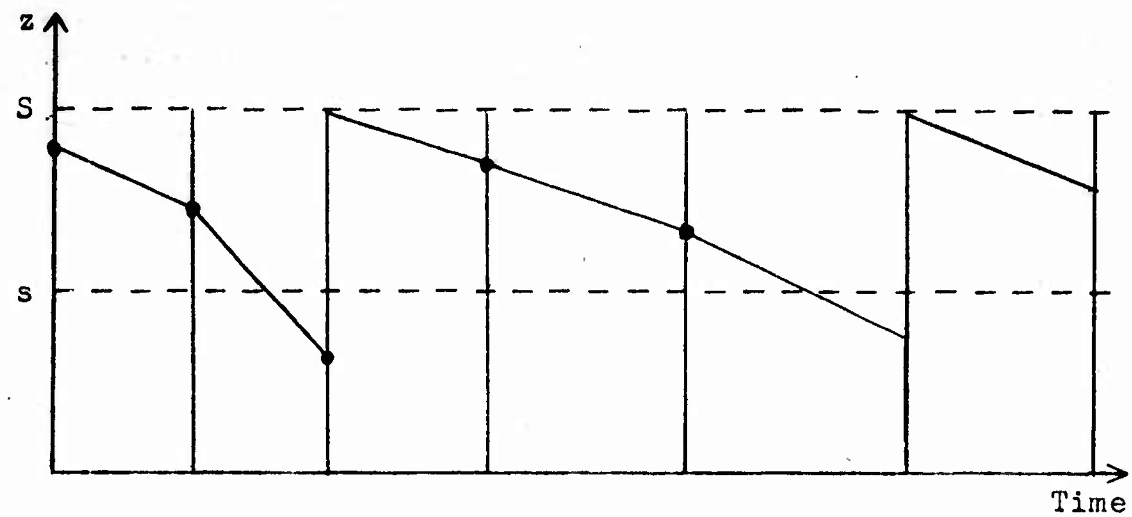

Figure 3

Figure 4 shows z_t as a function of y_t.

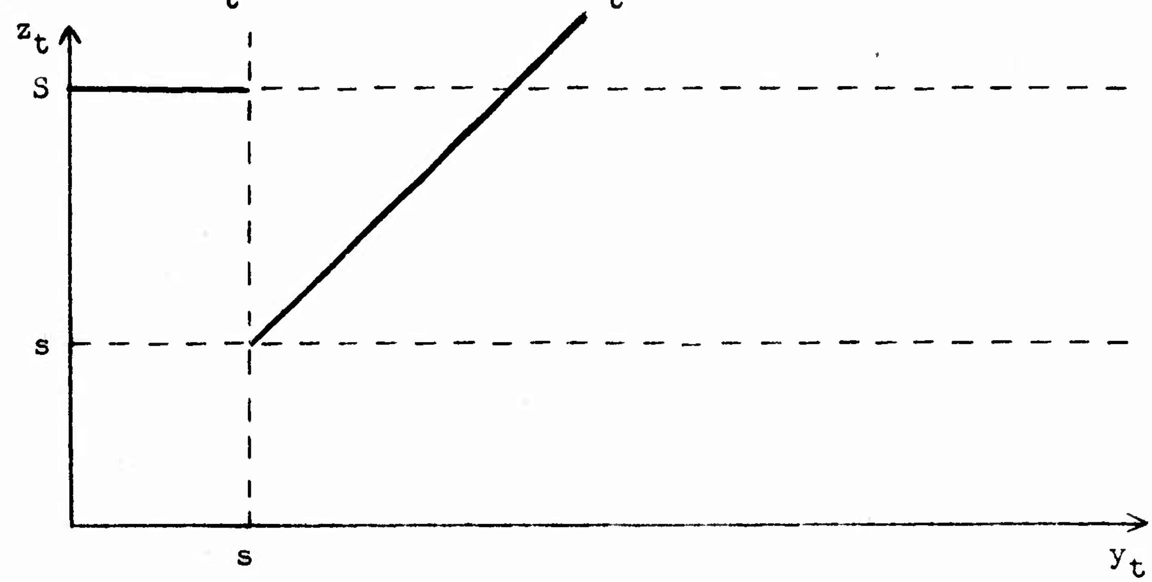

Figure 4

4:B We shall assume (as we have done in Sections 2 and 3) that the cost of handling an order does not depend on the amount ordered. Let this cost be K, a constant. Let the depletion penalty be A, a constant: compare (3.4), with $B = 0$. Let the variable cost of carrying a stock z_t during a unit of time be c, as in (3.4). Assume the purchasing price per unit of commodity to be independent of the amount bought, and equal to the marginal utility of one unit (i.e., in the notation of sub-section 3:A, $b_1 = 0$, $b_0 = a$). That is, the utility of operations of the agency, in excess of the expenses paid for these operations, is assumed constant, apart from the cost of storage and of handling orders. In the notations of 3:A, this constant is a_0, while K and c denote, respectively, the cost of handling an order (of any size) and the marginal cost of storage. Our assumption is an admissible approximation in the case of some non-profit agencies. It would be certainly both more general and more realistic to make the marginal utility of an operation differ from its purchasing price as was the case in our static model. But this will require further mathematical work (see 7:A).

4:C If y_0 is given and values S and s are chosen, the subsequent values y_t form a random process which is "Markovian"; see Feller [5], Chapter 15. That is, the probability distribution of y_{t+1}, given the value of y_t, is independent of $y_{t-1}, \cdots, y_0$. During the period $(t, t+1)$ a certain loss will be incurred whose conditional expectation, __for a fixed value of__ y_t, we denote by $\ell(y_t)$. Under the simplifying assumptions of sub-section 4:B,

$$(4.5)\qquad \ell(y_t) = \begin{cases} cy_t + A[1 - F(y_t)] & \text{for } y_t > s, \\ cS + A[1 - F(S)] + K & \text{for } y_t \le s . \end{cases}$$

Thus the function $\ell(y_t)$ involves S and s as parameters and is constant for $y_t \le s$. Note that

$$(4.6)\qquad \ell(0) = \ell(S) + K .$$

The unconditional expectation of the loss during $(t, t+1)$, that is, the expectation of $\ell(y_t)$, with y_t as a random variable, will be denoted by

$$(4.7)\qquad \ell_t = \ell_t(y_0) .$$

We shall write $\ell_t(y_0)$ rather than ℓ_t only when we need to emphasize the dependence of ℓ_t on the initial stock level. Clearly $\ell_0(y) = \ell(y)$ for every value y of y_0.

Figure 5 shows a possible type of graph for $\ell(y_t)$.

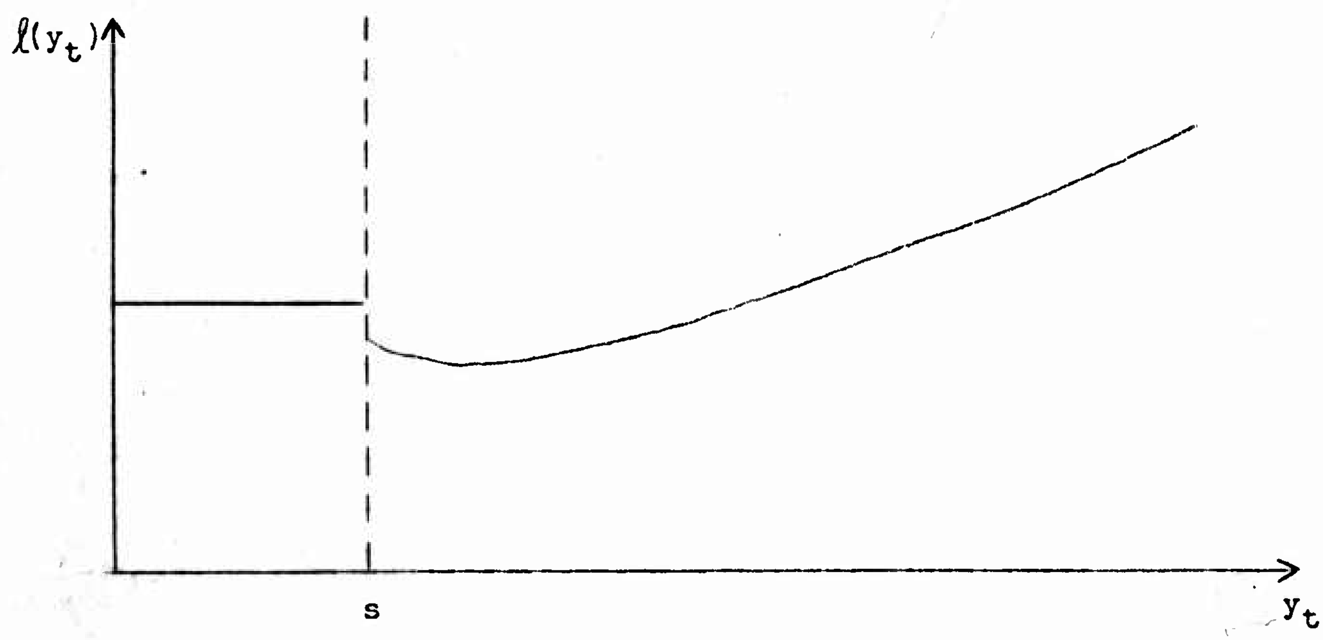

Figure 5

4:D We now introduce the concepts of a discount factor, α, and of "present value" of a loss. If the value of y_{t_0} is given, the present value at time t_0 of the expected loss incurred in the interval $(t_0 + t, t_0 + t + 1)$ is $\alpha^t E[\ell(y_{t_0+t})] = \alpha^t \ell_t(y_{t_0})$. When maximizing expected utility, the policymaker takes into account the "present values" of losses, not their values at the time when they are incurred. In commercial practice, α is equal to unity minus an appropriate market rate of interest. In non-profit practice, α would have to be evaluated separately. Later it will be shown, however (see Section 5:B), that, under certain conditions, the optimal values of the parameters S, s can be found for α essentially equal to 1.

If we now define the function

$$L(y) = \ell_0(y) + \alpha\ell_1(y) + \alpha^2\ell_2(y) + \cdots$$

we see from definition (4.7) that $L(y_t)$ is the present value at time t of the total expected loss incurred during the period (t, t + 1) and all subsequent periods when y_t is given. By definition L(y) involves the parameters S and s; and the policymaker fixes these parameters so as to minimize $L(y_0)$.

4:E Now suppose y_0 is given. For a fixed value of y_1, the present value of the total expected loss over all periods is

$$\ell(y_0) + \alpha\ell(y_1) + \alpha^2 E_{y_1}[\ell(y_2)] + \alpha^3 E_{y_1}[\ell(y_3)] + \cdots , \tag{4.8}$$

where we have used $E_{y_1}[\ell(y_r)]$ to denote the conditional expectation of

$\ell(y_r)$, given the fixed value y_1. Now

$$E_{y_1}[\ell(y_r)] = \ell_{r-1}(y_1), \qquad r = 1, 2, \cdots$$

because of the fact that if y_1 is fixed, the subsequent value y_r, $r = 1, 2, \cdots$, is connected with y_1 in the same manner that y_{r-1} is connected with y_0 if y_1 is not specified. Therefore the expression (4.8) is equal to

$$(4.9) \qquad \ell(y_0) + \alpha\ell_0(y_1) + \alpha^2\ell_1(y_1) + \alpha^3\ell_2(y_1) + \cdots =$$

$$\ell(y_0) + \alpha[\ell_0(y_1) + \alpha\ell_1(y_1) + \alpha^2\ell_2(y_1) + \cdots] =$$

$$\ell(y_0) + \alpha L(y_1) .$$

The total expected loss over all periods from the beginning, which by definition is $L(y_0)$, is the expectation of the expression in (4.9) with y_1 regarded as a random variable. Hence

$$(4.10) \qquad L(y_0) = \ell(y_0) + \alpha E[L(y_1)] .$$

To express the expected value of $L(y_1)$ as a function of y_0 we note that if $y_0 \leq s$, then $z_0 = S$ and $y_1 = \max(S - x_0, 0)$; while if $y_0 > s$, then $z_0 = y_0$ and $y_1 = \max(y_0 - x_0, 0)$. Thus

$$(4.10') \qquad E[L(y_1)] = \int_{0-}^{S} L(S-x)dF(x) + L(0)[1-F(S)] \quad \text{for } y_0 \leq s,$$

$$E[L(y_1)] = \int_{0-}^{y_0} L(y_0-x)dF(x) + L(0)[1-F(y_0)] \quad \text{for } y_0 > s.$$

(Notice that from the way we have defined the rule of action, $L(y)$ is constant for $0 \leq y \leq s$ so that $L(0)$ is unambiguously defined.) Putting $y_0 = y$ we obtain from (4.10) and (4.10') the equations

$$(4.11) \qquad L(y) = \ell(y) + \alpha\int_{0-}^{S} L(S-x)dF(x) + \alpha L(0)[1 - F(S)] \text{ if } y \leq s,$$

$$(4.12) \qquad L(y) = \ell(y) + \alpha\int_{0-}^{y} L(y-x)dF(x) + \alpha L(0)[1-F(y)] \text{ if } y > s.$$

Our problem is to find the function $L(y)$ that satisfies (4.11), (4.12); and to minimize $L(y)$ with respect to S, s.

It is not difficult to show that $L(y)$ and $\ell_t(y)$ for each t are measurable functions of y, but we leave such technicalities aside.

5. A Dynamic Model: Method of Solution

5:A In treating the equations (4.11) and (4.12) we drop for the time being the assumption that $F(x)$ has a density function and assume only that the random variable x cannot take negative values. In order to take care of the possibility that $F(x)$ has a discontinuity at $x = 0$ (i.e., a positive probability that $x = 0$) we adopt the convention that Stieltjes integrals of the form $\int_0^{(\ )}(\ )dF(x)$ will be understood to have $0-$ as the lower limit. We continue to assume that $\ell(y)$ is given by (4.5) but it is clear that a similar treatment would hold for any non-negative function $\ell(y)$ which is constant for $0 \leq y \leq s$ and satisfies certain obvious regularity conditions.

Since $\ell(y)$ and $L(y)$ are independent of y for $0 \leq y \leq s$, (4.11) tells us simply that

$$(5.1) \qquad L(0) = \ell(0) + \alpha\int_0^S L(S-x)dF(x) + \alpha L(0)\left[1 - F(S)\right],$$

while putting $y = S$ in (4.12) gives

$$(5.2) \qquad L(S) = \ell(S) + \alpha\int_0^S L(S-x)dF(x) + \alpha L(0)\left[1 - F(S)\right].$$

Subtracting (5.2) from (5.1) we obtain, using (4.6),

$$(5.3) \qquad L(0) - L(S) = K ,$$

an expression which is in fact obvious since if the initial stock is 0 we immediately order an amount S at a cost K for ordering. We shall solve the equation (4.12) for the function $L(y)$, considering

$L(0)$ as an unknown parameter, and then use (5.3) to determine $L(0)$.

On the right side of (4.12) we make the substitution

$$(5.4) \qquad \int_0^y L(y-x)\,dF(x) = \int_0^{y-s} L(y-s)\,dF(x) + L(0)\int_{y-s}^y dF(x)\ ;$$

the last term follows from the fact that $L(y-x) = L(0)$ when $0 \leq y-x \leq s$. Now make the change of variables

$$(5.5) \qquad y - s = \eta\ ,$$

$$L(y) = L(\eta + s) = \lambda(\eta)\ .$$

Putting (5.4) and (5.5) in (4.12) gives

$$(5.6) \qquad \lambda(\eta) = \ell(\eta + s) + \alpha L(0)\left[1 - F(\eta)\right] + \alpha\int_0^\eta \lambda(\eta - x)\,dF(x),\quad \eta > 0.$$

Equation (5.6) is in the standard form of the integral equation of renewal theory; see, for example, Feller's paper [4]. The solution of (5.6) can be expressed as follows. Define distribution functions $F_n(x)$, $n = 1, 2, \cdots$, (the convolutions of $F(x)$) by

$$(5.7) \qquad F_1(x) = F(x),$$

$$F_{n+1}(x) = \int_0^x F_n(x-u)\,dF(u)\ .$$

Define the function $H_\alpha(x)$

$$(5.8) \qquad H_\alpha(x) = \sum_{n=1}^\infty \alpha^n F_n(x), \qquad 0 \leq \alpha < 1.$$

It is not difficult to verify that the series in (5.8) converges for any value of α, less than 1 or not. Putting

$$(5.9) \qquad R(\eta) = \ell(\eta + s) + \alpha L(0)[1 - F(\eta)],$$

we can write the solution of (5.6) as

$$(5.10) \qquad \lambda(\eta) = R(\eta) + \int_0^\eta R(\eta - x)\,dH_\alpha(x)$$

$$= R(\eta) + \sum_{n=1}^{\infty} \alpha^n \int_0^\eta R(\eta - x)\,dF_n(x) \; .$$

In terms of L and ℓ, (5.10) gives

$$(5.11) \qquad L(y) = \ell(y) + \alpha L(0)[1 - F(y - s)]$$

$$+ \int_0^{y-s} \{\ell(y-x) + \alpha L(0)[1 - F(y-x-s)]\}\,dH_\alpha(x), \quad y > s.$$

From (5.3) and (5.11) we have

$$(5.12) \qquad L(0) - K = \ell(S) + \int_0^{S-s} \ell(S-x)\,dH_\alpha(x)$$

$$+ \alpha L(0)\left\{1 - F(S-s) + \int_0^{S-s} [1 - F(S-s-x)]\,dH_\alpha(x)\right\} .$$

In (5.12) we have a linear equation which we can solve for the unknown quantity L(0) which has, as we shall show, a nonvanishing coefficient in (5.12). This gives us the value of L(y) for $y \leq s$, and we can obtain L(y) for $y > s$ from (5.11), since every term on the right side of that equation is now known.

The coefficient of L(0) in (5.12) is

$$(5.13)\qquad 1-\alpha\left\{1-F(S-s)+\int_0^{S-s}\left[1-F(S-s-x)\right]dH_\alpha(x)\right\}$$

$$= 1-\alpha\left\{1-F(S-s)+H_\alpha(S-s)-\int_0^{S-s}F(S-s-x)\,dH_\alpha(x)\right\}$$

$$= 1-\alpha\left\{1-F(S-s)+\sum_{n=1}^{\infty}\alpha^n F_n(S-s)-\sum_{n=1}^{\infty}\alpha^n F_{n+1}(S-s)\right\}$$

$$= (1-\alpha)\left[1+H_\alpha(S-s)\right] .$$

Using (5.13) we obtain

$$(5.14)\qquad L(0) = \frac{K+\ell(S)+\int_0^{S-s}\ell(S-x)\,dH_\alpha(x)}{(1-\alpha)\left[1+H_\alpha(S-s)\right]} .$$

Knowing L(y) from (5.11) and (5.14), the next step is to find, for a given initial stock y_0, the values of s and S which minimize $L(y_0)$. We shall consider only the minimization of L(0), although the procedure could be worked out to minimize $L(y_0)$ for any initial stock y_0. The procedure of minimizing L(0) is not quite so special as it may appear. Suppose that for a given y_0 the values of s and S which minimize $L(y_0)$ are denoted by $s^*(y_0)$ and $S^*(y_0)$. If $s^*(0) > 0$ and if $s^*(y_0)$ and $S^*(y_0)$ are continuous functions of y_0 (a point which we have not investigated mathematically), then $s^*(y_0) = s^*(0)$ for sufficiently small y_0. To see this we write

$$L(y) = L(y;\, s,\, S)$$

to indicate the dependence of L on s and S. Let $I = [a, b]$ be a closed interval with $0 < a < s^*(0) < b$. Take y_0 small enough so that $s^*(y_0)$ is in I. Now if y_0 is sufficiently small

$$L\big(y_0;\ s^*(y_0),\ S^*(y_0)\big) = \min_{\substack{s \in I \\ S \geq s}} L(y_0;\ s,\ S) =$$

$$K + \min_{\substack{s \in I \\ S \geq s}} L(S;\ s,\ S),$$

which is independent of y_0, Q.E.D.

In 5:B an optimization criterion will be given which is independent of the initial level y_0.

We now re-introduce the assumption that F(x) has a probability density which is continuously differentiable,

$$F(x) = \int_0^x f(t)dt\ .$$

We recall from (4.5) that $\ell(y)$, for $y > s$, is given by

$$\ell(y) = A[1 - F(y)] + cy\ .$$

Consider the minimization of (5.14) with respect to s and S. First we consider the case where $S - s$ is fixed. The denominator of (5.14) involves S and s only as a function of $S - s$. We therefore have to minimize the numerator of (5.14) with respect to S, subject to the constraint that S is at least as great as the fixed value of $S - s$. If the minimum value does not occur for $S = S - s$ (i.e., $s = 0$), it occurs at a value of S for which the conditions

$$(5.15)\qquad c - Af(S) + \int_0^{S-s} [c - Af(S - x)]\, dH_\alpha(x) = 0 ,$$

$$(5.16)\qquad -Af'(S) - \int_0^{S-s} Af'(S - x)\, dH_\alpha(x) > 0,$$

hold. It should be noted that K does not enter into (5.15) and (5.16).

If we drop the requirement that $S - s$ be fixed, then s^* and S^*, provided they satisfy the condition $0 < s^* < S^*$, occur at a point where equation (5.15) holds, together with the equation obtained by setting the derivative of (5.14) with respect to $S - s$ equal to 0, taking the appropriate second-order conditions into account. We also need here the assumption that $H_\alpha(x)$ is the integral of a function $h_\alpha(x)$,

$$H_\alpha(x) = \int_0^x h_\alpha(t)\, dt .$$

Then differentiation of (5.14) with respect to $S - s$ gives, setting the derivative equal to 0,

$$(5.17)\qquad A[F(S) - F(s)] = c(S - s) + K + \int_0^{S-s} [c - Af(S-x)]\, H_\alpha(x)\, dx .$$

Presumably the minimization of (5.14) would be accomplished in practice by numerical methods.

5:B So far we have considered α as an arbitrary parameter. It is clear that if we let $\alpha \longrightarrow 1$, keeping s and S fixed, the quantity $L(0)$ becomes infinite. However, as we shall see, the quantity $(1 - \alpha)L(0)$ approaches a finite limiting value whose significance can be explained as follows. Suppose that levels s and S have been fixed and that y_0 is given. We have mentioned that the quantities y_t then form a Markovian random process. Moreover, the probability distribution of y_t, as $t \longrightarrow \infty$, approaches a fixed limiting distribution which is independent of y_0. (See Feller [5], Chapter 15, for the relevant theory when F(x) is a step-function. The general case is more difficult and the proof requires restrictions on F(x), which are, however, not of practical importance. See Doob [2].) This implies that ℓ_t, the expected loss in the interval (t, t + 1) approaches a limiting value ℓ_∞ which is independent of y_0. (The losses during successive time intervals form a sequence of bounded random variables.) As we shall see, we can find the value of ℓ_∞. Then if we do not want to use a discount factor α, one way to proceed is to pick s and S so as to minimize ℓ_∞. This is almost equivalent to minimizing the total expected loss over a long finite time interval.

Another way to look at the situation is as follows. The limiting distribution of y_t for large t is a "stationary distribution"; i.e., if y_0 has this distribution, instead of being fixed, then y_t has the same distribution for every t. The expected loss during (t, t + 1), if y_t has this distribution, is just ℓ_∞.

Since

$$L(0) = \ell_0(0) + \alpha\ell_1(0) + \alpha^2\ell_2(0) + \cdots ,$$

and $\ell_t(0) \longrightarrow \ell_\infty$ as $t \longrightarrow \infty$, we have

$$(5.18) \qquad L(0)(1-\alpha) = \ell_0(0) + \alpha[\ell_1(0) - \ell_0(0)] + \alpha^2[\ell_2(0) - \ell_1(0)] + \cdots .$$

The series

$$\ell_0(0) + [\ell_1(0) - \ell_0(0)] + [\ell_2(0) - \ell_1(0)] + \cdots$$

converges to the value ℓ_∞ and therefore, by a standard result of analysis, we have from (5.18),

$$\lim_{\alpha \to 1} L(0)(1 - \alpha) = \ell_\infty .$$

In order to determine ℓ_∞, we can then multiply the right side of (5.14) by $(1 - \alpha)$ and let $\alpha \to 1$, obtaining

$$(5.19) \qquad \ell_\infty = \frac{K + \ell(S) + \int_0^{S-s} \ell(S - x)dH(x)}{1 + H(S - s)}$$

where $H(x)$ is defined by

$$H(x) = \lim_{\alpha \to 1} H_\alpha(x) = \sum_{n=1}^{\infty} F_n(x) .$$

(It is not hard to see that the step

$$\lim_{\alpha \to 1} \int_0^{S-s} \ell(S - x)dH_\alpha(x) = \int_0^{S-s} \ell(S - x)dH(x)$$

is justified.)

We can then minimize the function in (5.19) with respect to s and S. It should be noted that ℓ_∞ is of course independent of the initial stock y_0.

6. A Dynamic Model: Examples

We consider now some examples for a particular function $F(x)$. It is advantageous to use a function whose convolutions can be written explicitly. From this point of view functions of the form

$$(6.1) \qquad F(x) = \frac{\beta^k}{(k-1)!} \int_0^x u^{k-1} e^{-\beta u} \, du, \qquad k > 0, \quad \beta > 0,$$

are convenient $\big((k-1)!$ is $\Gamma(k)$ if k is not an integer$\big)$ since by proper choice of β and k we can give any desired values to the mean and variance,

$$\bar{x} = k/\beta , \qquad \overline{x^2} - (\bar{x})^2 = k/\beta^2 ,$$

and since $F_n(x)$ is then given by

$$F_n(x) = \frac{\beta^{nk}}{(nk-1)!} \int_0^x u^{nk-1} e^{-\beta u} \, du .$$

The function $H_\alpha(x)$ is then given by

$$(6.2) \qquad H_\alpha(x) = \int_0^x e^{-\beta u} \left\{ \sum_{n=1}^{\infty} \frac{\beta^{nk} \alpha^n u^{nk-1}}{(nk-1)!} \right\} du .$$

If k is an integer the summation in (6.2) can be performed explicitly giving

$$(6.3) \qquad H_\alpha(x) = \frac{\beta \alpha^{1/k}}{k} \int_0^x e^{-\beta u} \left(\sum_{j=1}^{k} \omega_j \, e^{\omega_j \alpha^{1/k} \beta u} \right) du$$

where $\omega_1, \cdots, \omega_k$ are the k^{th} roots of unity. For example, if $k = 2$, we have $\omega_1 = -1$, $\omega_2 = 1$, so that

$$H_\alpha(x) = \frac{\beta\sqrt{\alpha}}{2}\int_0^x e^{-\beta u}\left(e^{\beta\sqrt{\alpha}\,u} - e^{-\beta\sqrt{\alpha}\,u}\right)du .$$

It is instructive to find the value of ℓ_∞ for the simple case $f(x) = e^{-x}$. In this case, from (6.3),

$$H(x) = \int_0^x e^{-u}(e^u)du = x$$

and we have

$$\ell_\infty = \frac{K + \ell(S) + \int_0^{S-s} \ell(S - x)dx}{1 + S - s} =$$

$$\frac{K + cS + Ae^{-S} + \int_0^{S-s}\left[c(S - x) + Ae^{-S+x}\right]dx}{1 + S - s} =$$

$$\frac{K + cS + Ae^{-S} + c(S - s) - \frac{c}{2}(S - s)^2 + Ae^{-S}(e^{S-s} - 1)}{1 + S - s} =$$

$$\frac{K + cS + Ae^{-s} + \frac{c}{2}(S^2 - s^2)}{1 + S - s} .$$

Letting $S - s = \Delta$, we see that this expression, for a fixed value of Δ, has its minimum (unless it occurs when $s = 0$) when

$$S = \log_e\left(\frac{A}{c}\right) - \log_e(1 + \Delta) + \Delta .$$

7. Further Problems and Generalizations

To make the dynamic model more realistic certain generalizations are necessary. We shall register them in the present section, as a program for further work.

7:A Of the several cost and utility parameters used in the certainty model of Section 2, and in the static uncertainty model of Section 3, we have retained in the dynamic uncertainty model only three: c, the marginal cost of storage; K, the constant cost of handling an order; and A, the constant part of the depletion penalty. We have thus dropped the parameters $\underline{\underline{a}}$, b_0, b_1, and B. The meaning of the first three of these was discussed in sub-section 4:B. It can be presumed from the equation (3.8) of the more developed static model that if we similarly developed the dynamic model, c could be easily replaced by $(c + b_0)$; but that $(B + a)$ would form an additional parameter, altogether excluded from our simple dynamic model. Difficulties of another kind will occur when $b_1 > 0$, i.e., when there are economies of big-lot buying, which are due not to the advantage of handling one order instead of many, but to the cheapness of transporting (and producing) large quantities. This will obviously modify the rule of action (4.4), as the loss that we intend to minimize will depend on $(S - y_t)$, the size of the replenishment order.

7:B Another direction in which the dynamic model must be generalized to become realistic, corresponds to sub-section 2:F of the certainty model. The cost of ordering may be a periodic function of time, due to existing schedules of transportation and administrative routine. In our dynamic model, orders (zero or positive) can be given at the beginning of each period of length $\theta_0 = 1$, the cost K of a positive order being constant. Suppose, however, that instead, $K = K^o$ at instants 0, $m\theta_0$, $2m\theta_0$, $\cdots$, where m is an integer; and $K = K^+$ at all other instants, with $K^+ > K^o$. For example, if θ_0 is one day, $m\theta_0$ may be one week, and instant 0 is the first Monday. Given the other parameters, it may or may not be advantageous to place orders on Mondays only. Moreover, the integer m may itself be a controlled variable: e.g., one may have to decide whether to make the orders daily, weekly, or monthly, a month not being an integral multiple of a week, and the cost of a monthly order being different from that of a weekly order.

7:C The aggregation problem, treated for the case of certainty in the last three sub-sections of Section 2, arises of course also in the case of uncertainty. The problem is important because the number of items handled by any large organization (excepting possibly some highly specialized ones) is usually very large; and, usually, only large organizations are equipped to implement an inventory policy approximating the optimal one, since it presupposes either a good knowledge of the relevant parameters, or their statistical estimation.

7:D We have assumed the distribution F(x) of demand per unit – period to be known – presumably estimated from previous samples. Actual estimations of this distribution were carried out by Fry for the Bell Telephone Company (see Wilson [10]), and by Kruskal [7] with the material of the medical branch of the U. S. Navy.

Inventory policies based on such estimates can be improved as the operations are going on. The logic of such "sequential" procedure can be outlined as follows:

Using the dynamic model of Section 4 with s and S as the controlled parameters, now permitted to vary with time, denote by $\mu = \mu[S(t), s(t); F]$ the expected loss over an infinite period, given the non-controlled parameters A and K, but with the distribution F(x) unknown. Denote by $X_t = (x_1, \ldots, x_{t-1})$ the sequence of past observations on demand (fulfilled or not). Find two functions $S = S_t\ (X_t)$ and $s = s_t\ (X_t)$ that would produce the best results. The best results can be defined as follows. The expected loss depends on the functions S_t, s_t and on the unknown distribution function F, and can be written as the functional on S, s, F,

$$\mu = \mu[S_t\ (X_t),\ s_t\ (X_t);\ F] = \mu(S,\ s;\ F)\ .$$

It is assumed that the distribution F belongs to some class specified in advance. The criterion for the most appropriate choice of policies in cases such as this are still a matter of dispute. One suggestion, inspired by Wald's statistical writings, is that Nature should be visualized as having chosen F so as to maximize the loss; the aim of the inventory controller — or any other planner — should then, in accordance with the theory of games, be to minimize the maximum loss. Professor Leonard J. Savage has pointed out that it would be better to regard the penalty to be assessed against the planner as not the total loss but only that part which is due to his ignorance, the "regret". This is defined as the difference between the minimum expected loss in the case tion are its opponents and have the choice of demands x_t, within a certain range. If they are required to choose F at the beginning of the operation, then the proper rule is to choose the functions S, s so as to minimize

$$\max_{F} \lambda(S, s; F).$$

However, there are other possibilities. For example, the opponents might be permitted by the rules of the game to choose x_t at each time t. Then the specification of the game will have to be completed by a statement of the enemy's information pattern and the costs to him, if any, in choosing the various values of x_t.

that the organization knew the distribution F, and the expected loss resulting from its actual decision. Given F, the minimum expected loss is

$$\min_{S,s} \mu(S,s;\ F) = \mu^*(F), \quad \text{say} .$$

The "regret" is

$$r = \mu(S,s;\ F) - \mu^*(F) = r(S,s;\ F), \text{ say.}$$

The optimal stock control functions $S_t^*(X_t)$, $s_t^*(X_t)$ and the best estimator of the distribution $F^* = F_t^*(x,\ X_t)$ must satisfy the condition

$$r(S^*,\ s^*\ ;\ F^*) = \min_{S,s} \max_{F} r(S,\ s;\ F) .$$

Another proposal, much more traditional in probability theory, is to assume that the planner has some psychological probability distribution G over all possible distributions F, representing his relative degrees of belief in their occurrences. The distribution G reflects past experience and general judgment. Then the optimal procedure is to choose the functions S and s so as to minimize the expected loss,

$$\int \lambda(S,s;\ F)dG .$$

Whether some kind of minimax criterion is adopted or a psychological probability approach is taken, it is easy to see that the solution will call for functions S, s which really depend on the observations X_t. However, the determination of these functions is still an unsolved problem.

Another case is that in which the "customers" of the organiza-

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